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Approximate solutions and optimality conditions of vector variational inequalities in Banach spaces

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Abstract In this paper, we introduce and discuss the notion of ε-solutions of vector variational inequalities. Using convex analysis and nonsmooth analysis, we provide some sufficient conditions and necessary conditions for a point to be an ε-solution of vector variational inequalities.

Keywords Vector variational inequality · Approximate solution · Optimality condition

Mathematics Subject Classification (2000) 58E35 · 58E17 · 90C46

1 Introduction

The vector variational inequality model has attracted extensive attention in recent years. The edited book [3] has included survey papers and research papers on this topic. The study of this model has been motivated by its applications in vector optimization and vector traffic equilibrium problems, see Ref. [3].

Optimality conditions for a solution of vector variational inequalities have been obtained in Ref. [7,8]. More precisely, a necessary and sufficient condition is obtained in Ref. [8] by converting a vector variational inequality problem to an equivalent vector optimization problem and assuming some convexity condition and a necessary optimality condition is obtained in Ref. [7] using a generalized directional derivative and a pseudo-convexity condition. An approximate vector variational inequality problem, or an ϵ-solution of a vector variational inequality problem, has been introduced and used to describe an approximate Pareto solution

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of vector optimization problems in Ref. [1]. In terms of an ϵ-subgradient, relations between an ϵ-solution of a vector variational inequality problem and an approximate Pareto solution of a vector optimization problem are discussed in Ref. [6].

In this paper, we investigate optimality conditions for the notion of ε-solutions of vector variational inequalities. Our study is based on convex analysis and nonsmooth analysis. We obtain sufficient conditions for ε-solutions of vector variational inequality problems in terms of a shifted gap function in the dual form. Under a generalized convexity of the feasible set, we show that one of these sufficient conditions is also necessary. We also obtain various kinds of sufficient or necessary optimality conditions using the Clarke normal cone either in the space of feasible solutions or in the space of objective values. We give a counter example to show that the convexity of the feasible set is necessary to guarantee the validness of a sufficient condition.

2 Preliminaries

Let X be a Banach space and A be a closed subset of X with $a \in A$. Let $T(A, a)$ denote the Clarke tangent cone of A at a which is defined by

$$T(A, a) = \liminf_{x \overset{A}{\to} a, t \to 0^+} \frac{A - x}{t},$$

where $x \overset{A}{\to} a$ means that $x \to a$ with $x \in A$. Thus, $v \in T(A, a)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all n. Let $N(A, a)$ denote the Clarke normal cone, that is,

$$N(A, a) = \{x^* \in X^* : \langle x^*, h\rangle \leq 0 \quad \forall h \in T(A, a)\}.$$

It is well known that if A is convex then

$$N(A, a) = \{x^* \in X^* : \langle x^*, x - a\rangle \leq 0 \quad \forall x \in A\}.$$

Let $\phi : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous function. For $x_0 \in \mathrm{dom}(\phi)$, let $\partial\phi(x_0)$ denote the Clarke subdifferential of ϕ at x_0. It is known (cf [2]) that

$$\partial\phi(x_0) = \{x^* \in X^* : (x^*, -1) \in N(\mathrm{epi}(\phi), (x_0, \phi(x_0)))\},$$

where $\mathrm{epi}(\phi) = \{(x, t) \in X \times R : \phi(x) \leq t\}$. When ϕ is convex, one has

$$\partial\phi(x_0) = \{x^* \in X^* : \langle x^*, x - x_0\rangle \leq \phi(x) - \phi(x_0) \quad \forall x \in X\}.$$

We will need the following known result (see [[2], P.53, Corollary]).

Proposition 2.1 *Let ϕ be locally Lipschitz on X and $a \in A$. Suppose that $\phi(a) = \inf\{\phi(x) : x \in A\}$. Then $0 \in \partial\phi(a) + N(A, a)$.*

Let Y be another Banach space and $C \subset Y$ be a closed convex cone with a nonempty interior. The cone C induces the following ordering relationships in Y:

$$y_1 \leq_C y_2 \Longleftrightarrow y_2 - y_1 \in C,$$
$$y_1 \leq_{intC} y_2 \Longleftrightarrow y_2 - y_1 \in \mathrm{int}(C),$$
$$y_1 \not\leq_{C\setminus\{0\}} y_2 \Longleftrightarrow y_2 - y_1 \notin C \setminus \{0\},$$
$$y_1 \not\leq_{intC} y_2 \Longleftrightarrow y_2 - y_1 \notin \mathrm{int}(C).$$

Let $F : X \to L(X, Y)$ be a mapping, where $L(X, Y)$ denotes the set of all continuous linear operators from X to Y. Consider the following vector variational inequality problems

$$\text{(WVVI)} \qquad \text{find } x \in A \text{ such that } F(x)(z-x) \not\leq_{intC} 0 \text{ for all } z \in A,$$

and

$$\text{(VVI)} \qquad \text{find } x \in A \text{ such that } F(x)(z-x) \not\leq_{C\setminus\{0\}} 0 \text{ for all } z \in A$$

It is clear that $\bar{x} \in A$ is a solution of (WVVI) (resp. (VVI)) if and only if

$$F(\bar{x})(A-\bar{x}) \in Y \setminus -\text{int}(C) \quad (\text{resp. } F(\bar{x})(A-\bar{x}) \in Y \setminus -(C \setminus \{0\})).$$

This gives rise to the following notion of ε-solutions. For $\varepsilon \geq 0$, we say that $\bar{x} \in A$ is an ε-solution of (WVVI) (resp. (VVI)) if

$$F(\bar{x})(A-\bar{x}) \subset Y \setminus -\text{int}(C) + \varepsilon B_Y \quad (\text{resp. } F(\bar{x})(A-\bar{x}) \subset Y \setminus -(C \setminus \{0\}) + \varepsilon B_Y),$$

where B_Y denotes the unit ball of Y.

3 Main results

Throughout this section, we assume that e_0 is a fixed point in $\text{int}(C)$ with $\|e_0\| = 1$. Let C^+ denote the dual cone of C, that is, $C^+ = \{c^* \in Y^* : \langle c^*, c\rangle \geq 0 \quad \forall c \in C\}$, and let

$$C_1^+ := \{c^* \in C^+ : \langle c^*, e_0\rangle = 1\}.$$

It is clear that

$$\|c^*\| \geq 1 \quad \forall c^* \in C_1^+. \tag{1}$$

Moreover, C_1^+ is a weak*-comapct convex set in Y^*. Indeed, it is clear that C_1^+ is a weak*-closed convex subset of Y^*. In view of the Alaoglu Theorem, to prove the weak*-compactness of C_1^+, we need only show that C_1^+ is bounded. Take $r > 0$ such that $e_0 + rB_Y \subset C$ (by $e_0 \in \text{int}(C)$). It follows that, for any $c^* \in C_1^+$,

$$1 - r\|c^*\| = \inf_{c \in e_0 + rB_Y} \langle c^*, c\rangle \geq 0.$$

This shows that C_1^+ is bounded.

In the remainder of this section, we always assume that the ordering cone C is nontrivial, that is, $C \neq X$. Thus, $0 \notin \text{int}(C)$. It follows that $C_1^+ \neq \emptyset$.

We first provide a sufficient condition for a point in A to be an ε-solution of (WVVI) and (VVI).

Proposition 3.1 *Let A be a closed subset of X, $a \in A$ and $\varepsilon \geq 0$. We have*

(i) If there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that

$$\inf\{\langle F(a)^*(c^*), x\rangle : x \in A\} \geq \langle F(a)^*(c^*), a\rangle - \varepsilon, \tag{2}$$

where $F(a)^*$ denotes the conjugate operator of $F(a)$, then, for any $r > 1$, a is an $r\varepsilon$-solution of (WVVI).

(ii) If there exists $c^* \in intC^+$ with $\|c^*\| = 1$ such that (2) holds, then, for any $r > 1$, a is an $r\varepsilon$-solution of (VVI).

Proof

(i) First suppose that $\varepsilon = 0$. In this case, (2) means that $\langle c^*, F(a)(x - a)\rangle \geq 0$ for all $x \in A$. It follows from $c^* \in C^+ \setminus \{0\}$ that $F(a)(x - a) \not\leq_{intC} 0$ for all $x \in A$. Hence a is a solution of (WVVI).

Next we consider the case when $\varepsilon > 0$. Suppose to the contrary that there exists $r > 1$ such that a is not an $r\varepsilon$-solution of (WVVI). Then there exists $\tilde{a} \notin A$ such that $F(a)(\tilde{a} - a) \notin Y \setminus -\text{int}(C) + r\varepsilon B_Y$. It follows that $F(a)(\tilde{a} - a) + r\varepsilon B_Y \subset -\text{int}(C)$. Hence,

$$\langle c^*, F(a)(\tilde{a} - a)\rangle + \|c^*\|r\varepsilon = \sup\{\langle c^*, y\rangle : \ y \in F(a)(\tilde{a} - a) + r\varepsilon B_Y\} \leq 0.$$

This and $\|c^*\| = 1$ imply that

$$\langle F(a)^*(c^*), \tilde{a}\rangle \leq \langle F(a)^*(c^*), a\rangle - r\varepsilon < \langle F(a)^*(c^*), a\rangle - \varepsilon,$$

contradicting (2).

(ii) This can be proved similarly by noting the fact that $\langle c^*, b\rangle \geq 0$ and $c^* \in intC^+$ imply $b \not\leq_{C\setminus\{0\}} 0$. The proof is completed.

We make a note that in the special case when $X = Y = R^n$, $C = R^n_+$, $\varepsilon = 0$ and A is convex, Proposition 3.1 (i) reduces to Theorem 1 in Ref. [4].

Under some generalized convexity assumption of A, we can establish a necessity result. To do this, we will need the following lemma.

Lemma 3.1 *Let* $D := \{y \in Y : \ (y + B_Y) \subset -\text{int}(C)\}$ *and* $r_0 := d(0, D)$. *Then* D *is a convex set with a nonempty interior and* $r_0 \geq 1$.

Proof Let $y_1, y_2 \in D$ and $t \in [0, \ 1]$. Then $y_1 + B_Y \subset -\text{int}(C)$ and $y_2 + B_Y \subset -\text{int}(C)$. It follows from the convexity of C that

$$ty_1 + (1 - t)y_2 + B_Y = t(y_1 + B_Y) + (1 - t)(y_2 + B_Y) \subset -\text{int}(C).$$

Hence D is convex. Since $e_0 \in \text{int}(C)$ and $\|e_0\| = 1$, there exists $\delta > 0$ such that $e_0 + 2\delta B_Y \subset C$, and so $e_0 + \delta B_Y \subset \text{int}(C)$. Hence $-\frac{e_0}{\delta} + B_Y = -\frac{1}{\delta}(e_0 + \delta B_Y) \subset -\text{int}(C)$. This implies that $-\frac{e_0}{\delta} \in D$. Noting that $D - \text{int}(C) \subset D$, it follows that D has a nonempty interior. It remains to show that $r_0 \geq 1$. Suppose to the contrary that $r_0 < 1$. Then there exists $y_0 \in D$ such that $\|y_0\| < 1$. This and the definition of D imply that $0 \in -\text{int}(C)$. Since C is a cone, $C = X$, a contradiction. The proof is completed.

Let $A \subset X$ and $f : X \to Y$. f is said to be cone convex-like on A if, for any $b_1, b_2 \in A$, and any $t \in [0, \ 1]$, there exists $b \in A$ such that

$$f(b) \leq_C \ f(tb_1 + (1 - t)b_2).$$

It is known that f is cone convex-like on A if and only if the set $f(A) + C$ is convex, see Ref. [5].

Proposition 3.2 *Let* $\varepsilon \geq 0$ *and* $a \in A$. *Suppose that* $a \in A$ *is an* ε*-solution of (WVVI) and that* $F(a)$ *is cone convex-like on* A. *Then there exists* $c^* \in C^+$ *with* $\|c^*\| = 1$ *such that*

$$\inf\{\langle F(a)^*(c^*), x\rangle : \ x \in A\} \geq \langle F(a)^*(c^*), a\rangle - r_0\varepsilon, \tag{3}$$

where r_0 *is as in Lemma 3.1.*

Proof First we consider the case when $\varepsilon = 0$. In this case, a is a solution of (WVVI). Hence $F(a)(A-a) \subset Y \setminus -\text{int}(C)$, that is, $F(a)(A-a) \cap -\text{int}(C) = \emptyset$. Noting that $\text{int}(C) + C \subset \text{int}(C)$, this implies that $(F(a)(A-a)+C) \cap -\text{int}(C) = \emptyset$. Noting that, by the definition of cone convex-likeness, $F(a)(A-a)+C$ is a convex subset of Y, it follows from the separation theorem that there exists $c^* \in Y^*$ with $\|c^*\| = 1$ such that

$$\inf\{\langle c^*, F(a)(x-a)+c\rangle : (x,c) \in A \times C\} \geq \sup\{\langle c^*, y\rangle : y \in -C\}.$$

Noting that C is a cone, it follows that $c^* \in C^+ \setminus \{0\}$ and (3) holds.

Next suppose that $\varepsilon > 0$. Let $D_\varepsilon := \varepsilon D$, where D is as in Lemma 3.1. Then D_ε is a convex set with a nonempty interior. We claim that

$$F(a)(A-a) \cap D_\varepsilon = \emptyset. \tag{4}$$

Granting this and noting that $D_\varepsilon - C = \varepsilon(D-C) \subset \varepsilon D = D_\varepsilon$, it follows that $(F(a)(A-a) + C) \cap D_\varepsilon = \emptyset$. By the separation theorem, there exists $c^* \in Y^*$ with $\|c^*\| = 1$ such that

$$\inf\{\langle c^*, y\rangle : \in F(a)(A-a)+C\} \geq \sup\{\langle c^*, y\rangle : y \in D_\varepsilon\}.$$

This implies that $c^* \in C^+$ and

$$\inf\{\langle c^*, y\rangle : \in F(a)(A-a)\} \geq \sup\{\langle c^*, y\rangle : y \in D_\varepsilon\}. \tag{5}$$

On the other hand, by the definition of r_0, there exists a sequence $\{y_n\}$ in D such that $\|y_n\| \to r_0$. Hence $\langle c^*, \varepsilon y_n\rangle \geq -\|c^*\|\varepsilon\|y_n\| = -\varepsilon\|y_n\| \to -\varepsilon r_0$. This and (5) imply that (3) holds. It remains to show that (4) holds. Suppose to the contrary that there exists $y \in D_\varepsilon \cap F(a)(A-a)$. Then, $\frac{y}{\varepsilon} \in D$, that is, $\frac{y}{\varepsilon} + B_Y \subset -\text{int}(C)$. Since C is a cone, $y \subset -\text{int}(C) + \varepsilon B_Y$. Hence $y \notin Y \setminus -\text{int}(C) + \varepsilon B_Y$, contradicting the assumption that a is an ε-solution of (WVVI). The proof is completed.

The following corollary is immediate from Propositions 3.1 and 3.2.

Corollary 3.1 *Suppose that $a \in A$ and $F(a)$ is cone convex-like on A. Then a is a solution of (WVVI) if and only if there exists $c^* \in C_1^+$ such that*

$$\langle F(a)^*(c^*), a\rangle = \inf\{\langle F(a)^*(c^*), x\rangle : x \in A\}. \tag{6}$$

Remark Corollary 3.1 means that (WVVI) is solvable if and only if there exists $c^* \in C_1^+$ such that the following scalar variational inequality

$$\text{(SVI)} \quad \text{find } x \in A \text{ such that } \langle F(x)^*(c^*), z-x\rangle \geq 0 \text{ for all } z \in A$$

is solvable.

Proposition 3.3 *Let $L \in (\text{diam}(A), +\infty)$ and $\varepsilon \geq 0$. Let A be a closed convex subset of X and $a \in A$. Suppose that*

$$0 \in F(a)^*(C_1^+) + N(A,a) + \frac{\varepsilon}{L}B_{X^*}. \tag{7}$$

Then a is an ε-solution of (WVVI).

Proof First suppose that $\varepsilon = 0$. In this case, (7) means that there exists $c^* \in C_1^+$ such that $-F(a)^*(c^*) \in N(A,a)$. It follows from the convexity of A that $\inf\{\langle F^*(a)(c^*), x\rangle : x \in A\} = \langle F(a)^*(c^*), a\rangle$. This and Corollary 3.1 imply that a is a solution of (WVVI).

Next we consider the case when $\varepsilon > 0$. Suppose to the contrary that a is not an ε-solution of (WVVI). Then there exists $\tilde{a} \notin A$ such that $F(a)(\tilde{a} - a) \notin Y \setminus -\text{int}(C) + \varepsilon B_Y$. It follows that

$$F(a)(\tilde{a} - a) + \varepsilon B_Y \subset -\text{int}(C). \tag{8}$$

By (7), there exist $c^* \in C_1^+$ and $u^* \in B_{X^*}$ such that $x^* = -F(a)^*(c^*) + \frac{\varepsilon}{L}u^* \in N(A, a)$. Thus,

$$\langle -F(a)^*(c^*) + \frac{\varepsilon}{L}u^*, x - a\rangle \leq 0 \quad \forall x \in A.$$

It follows that

$$\langle c^*, F(a)(\tilde{a} - a)\rangle = \langle F(a)^*(c^*), \tilde{a} - a\rangle \geq \langle \frac{\varepsilon}{L}u^*, \tilde{a} - a\rangle \geq -\frac{\varepsilon}{L}\|\tilde{a} - a\|.$$

This and the choice of L imply that

$$\langle c^*, F(a)(\tilde{a} - a)\rangle > -\varepsilon. \tag{9}$$

On the other hand, by (8) one has

$$\langle c^*, F(a)(\tilde{a} - a)\rangle + \varepsilon\|c^*\| = \sup\{\langle c^*, y\rangle : \ y \in F(a)(\tilde{a} - a) + \varepsilon B_Y\} \leq 0.$$

This and (1) imply that

$$\langle c^*, F(a)(\tilde{a} - a)\rangle \leq -\varepsilon\|c^*\| \leq -\varepsilon,$$

contradicting (9). The proof is completed.

The following example shows that even in the special case when $\varepsilon = 0$, Proposition 3.3 does not hold if the convexity assumption of A is dropped.

Example 3.1 Let $X = Y = R^2$ and $C = \{(s, t) \in R^2 : \ t \geq 0\}$. Let $F(x)$ be the identical mapping on R^2 for all $x \in X$ and $A = \{(s, t) \in R^2 : \ s^2 + t^2 \leq 2 \text{ and } s^3 \leq t\}$. Take $e_0 = (0, 1)$ and $a = (0, 0)$. Then $C_1^+ = \{(0, 1)\}$. We claim that

$$N(A, (0, 0)) = \{(0, t) : \ t \leq 0\}. \tag{10}$$

Granting this and noting that $N(F(a)(A), F(a)) = N(A, (0, 0))$, we have that (7) holds with $\varepsilon = 0$. But, it is clear that

$$F(a)(A - a) \cap -\text{int}(C) = A \cap -\text{int}(C) \ni (-1, -1).$$

This implies that $F(a)(A - a)$ is not a subset $Y \setminus -\text{int}(C)$. Thus, a is not a solution of (WVVI). Next, we show that (10) holds. To do this, it suffices to show that

$$T(A, (0, 0)) = \{(s, t) \in R^2 : \ t \geq 0\}. \tag{11}$$

Let $(u, v) \in T(A, (0, 0))$ and take sequences $x_n \to 0$ and $t_n \searrow 0^+$. Then there exists $(u_n, v_n) \to (u, v)$ such that $(x_n, x_n^3) + t_n(u_n, v_n) \in A$ for any n. Hence,

$$(x_n + t_n u_n)^3 \leq x_n^3 + t_n v_n \quad \forall n.$$

This means that $3x_n^2 u_n + 3x_n t_n u_n^2 + t_n^2 u_n^3 \leq v_n$ for any n. Letting $n \to \infty$, one has $0 \leq v$. Hence $T(A, (0, 0)) \subset \{(s, t) \in R^2 : \ t \geq 0\}$. To prove the converse inclusion, let $(u, v) \in R^2$ with $v \geq 0$. Take any sequence $\{(x_n, y_n)\}$ in A with $(x_n, y_n) \to (0, 0)$ and any sequence

$\{t_n\}$ with $t_n \searrow 0$. For every natural number n, let $v_n := 3x_n^2 u + 3x_n t_n u^2 + t_n^2 u^3 + v$. Then $(u, v_n) \to (u, v)$ and for all n

$$\begin{aligned}(x_n + t_n u)^3 &= x_n^3 + 3x_n^2 t_n u + 3x_n t_n^2 u^2 + t_n^3 u^3 \\ &\le y_n + t_n(3x_n^2 u + 3x_n t_n u^2 + t_n^2 u^3) \le y_n + t_n v_n.\end{aligned}$$

On the other hand, $(x_n, y_n) \to (0, 0)$ and $t_n \searrow 0$ imply that there exists a natural number n_0 such that $(x_n + t_n u)^2 + (y_n + t_n v_n)^2 \le 2$ for all $n > n_0$. Let $(\tilde{u}_n, \tilde{v}_n) = (0, 0)$ for all $n \le n_0$ and $(\tilde{u}_n, \tilde{v}_n) = (u, v_n)$ for all $n > n_0$. Then $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ and $(x_n, y_n) + t_n(u, v_n) \in A$ for all n. This implies that $(u, v) \in T(A, (0, 0))$. Hence $T(A, (0, 0)) \supset \{(s, t) \in R^2 : \ t \ge 0\}$. This shows that (11) holds.

Dropping the convexity of A, we have the following necessity result.

Proposition 3.4 *Let A be a closed subset of X and suppose that $a \in A$ is a solution of (WVVI). Then*

$$0 \in F(a)^*(C_1^+) + N(A, a). \tag{12}$$

Proof Since $e_0 \in \text{int}(C)$, it is easy to verify that $e_0 - \text{int}(C)$ is an open convex neighborhood of 0 and e_0 is on its boundary. Let P denote the Minkowski function of $e_0 - \text{int}(C)$, that is,

$$P(y) = \inf\{t > 0 : \ y \in t(e_0 - \text{int}(C))\} \quad \forall y \in Y.$$

Then

$$P(e_0) = 1 \ \text{ and } \ e_0 - \text{int}(C) = \{y \in Y : \ P(y) < 1\}. \tag{13}$$

We claim that

$$\partial P(e_0) \subset C_1^+. \tag{14}$$

Let $y^* \in \partial P(e_0)$. Then, $\langle y^*, y - e_0\rangle \le P(y) - P(e_0)$ for any $y \in Y$. This and (13) imply that $\langle y^*, -e_0\rangle \le P(0) - P(e_0) = -1$ and

$$\langle y^*, -c\rangle \le P(e_0 - c) - P(e_0) \le 0 \quad \forall c \in \text{int}(C).$$

It follows that $y^* \in C^+$. On the other hand, $\langle y^*, e_0\rangle \le P(2e_0) - P(e_0) = P(e_0) = 1$. Therefore, $y^* \in C_1^+$. This shows that (14) holds. Since a is a solution of (WVVI). Then $F(a)(A - a) \cap -\text{int}(C) = \emptyset$. Hence,

$$(e_0 + F(a)(A - a)) \cap (e_0 - \text{int}(C)) = \emptyset.$$

This and (13) imply that

$$P(e_0) = \inf\{P(y) : \ y \in e_0 + F(a)(A - a)\}. \tag{15}$$

Let $f(x) := P(e_0 + F(a)(x - a))$ for all $x \in X$. Then f is a continuous convex function on X and $f(a) = \inf\{f(x) : \ x \in A\}$. It follows from Proposition 2.1 that

$$0 \in \partial f(a) + N(A, a). \tag{16}$$

Since the bounded linear operator $F(a) : X \to Y$ is strictly differentiable, [[2], Theorem 2.3.10] implies that $\partial f(a) \subset F(a)^*(\partial P(e_0))$. It follows from (14) that (12) holds. The proof is completed.

Proposition 3.5 *Let A be a closed subset of X and suppose that $a \in A$ is a solution of (WVVI). Then*

$$0 \in C_1^+ + N(F(a)(A), F(a)(a)). \tag{17}$$

Proof As in the proof of Proposition 3.4, (15) holds. It follows from Proposition 2.1 that $0 \in \partial P(e_0) + N(e_0 + F(a)(A - a), e_0)$. Noting that

$$N(e_0 + F(a)(A - a), e_0) = N(F(a)(A), F(a)(a)),$$

this and (14) imply that (17) holds.

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References

1. Chen, G.-Y., Craven, B.D.: Approximate dual and approximate vector variational inequality for multiobjective optimization. J. Austral. Math. Soc. Ser. A **47**, 418–423 (1989)
2. Clarke, F.H.: Optimization and Nonsmooth Analysis. Les publications CRM, Montreal, Canada (1989)
3. Giannessi, F.: Vector Variational Inequalities and Vector Equilibria. Mathematical theories. Nonconvex Optimization and its Applications, 38. Kluwer Academic Publishers, Dordrecht (2000)
4. Goh, C.J., Yang, X.Q. : On scalarization methods for vector variational inequalities. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria, pp. 217–232. Kluwer Academic Publishers, Dordrecht/Boston/London (2000)
5. Jeyakumar, V.: Convexlike alternative theorems and mathematical programming. Optimization **16**, 643–652 (1985)
6. Rong, W.D.: Epsilon-approximate solutions to vector optimization problems and vector variational inequalities. (Chinese) Nei Monggol Daxue Xuebao Ziran Kexue **23**, 513–518 (1992)
7. Ward, D.E., Lee, G.M.: On relations between vector optimization problems and vector variational inequalities. J. Optim. Theory Appl. **113**, 583–596 (2002)
8. Yang, X.Q.: Generalized convex functions and vector variational inequalities. J. Optim. Theory Appl. **79**, 563–580 (1993)